Teleparallel equivalent of general relativity: a critical review

Janusz Garecki[1]

Institute of Mathematics, University of Szczecin, Wielkopolska 15, 70-451 Szczecin, Poland

This work is a slightly amended lecture delivered at **Hypercomplex Seminar 2010** dedicated to Professor Roman S. Ingarden on the occasion of the Jubilee of His 90th birthday, 17-24 July 2010, at Będlewo, Poland

Abstract

After reminder some facts concerning general relativity (**GR**) we pass to *teleparallel gravity*. We are confining to the special model of the teleparallel gravity, which is popular recently, called *the teleparallel equivalent of general relativity* (**TEGR**). We are finishing with conclusion and some general remarks.

---

[1]E-mail address: garecki@wmf.univ.szczecin.pl

1 Introduction and standard formulation of GR

As it is known **GR** is a modern geometrical theory of gravity which simultaneously gives a mathematical model of the physical spacetime.

The mathematical model of the physical spacetime in **GR** is given by a pseudo-Riemannian differential manifold (Haussdorff, paracompact, connected, inextensible, orientable) (M_4, g_L). Here g_L means a Lorentzian metric which satisfies *Einstein equations*

$$G_\mu{}^\nu = \frac{8\pi G}{c^4} T_\mu{}^\nu \tag{1}$$

$(\alpha, \ \beta, \ \gamma, ..., \mu, \ \nu, ..., = 0, 1, 2, 3)^2$

So, g_L, is a *dynamical object*.

Here $G_\mu{}^\nu$ is the so-called *Einstein tensor*, $T_\mu{}^\nu$ is the *matter energy-momentum tensor* (the source of the gravitational field), c is the velocity of light in vacuum, and G means Newtonian gravitational constant.

The mathematical model of the physical spacetime in **GR** originated from *Einstein Equivalence Principle* (**EEP**)[1]. The main ingredient of this Principle is *universality of the free falls* of the test bodies in a given gravitational field.

GR reduces the gravitational interactions to some geometric aspects of the spacetime. Namely, we have:

1. g_L = gravitational potentials,

2. $\{^\alpha_{\beta\ \gamma}\}$ = gravitational strengths, and

3. $R^\alpha{}_{\beta\gamma\delta}(\{\})$ = strengths of the gravitational tidal forces.

The symmetry group of the **GR** is the infinite group **DiffM$_4$**.

The Levi-Civita connection $\{^\alpha_{\beta\ \gamma}\}$ is symmetric, metric and torsion-free.

Usually one uses in **GR** a maximal atlas of the local charts (local maps, coordinate patches) and *implicite* coordinate frames (natural frames, holonomic frames) and coframes ($\{\partial_\mu\}$, $\{dx^\alpha\}$) and coordinate components of the geometrical objects.

---

[2] We will identify geometrical objects with the sets of their components. Greek indices mean *coordinate components* of the geometrical objects.

Every coordinate transformation

$$x^{\alpha'} = x^{\alpha'}(x^{\beta}), \ det[\frac{\partial x^{\alpha'}}{\partial x^{\beta}}] \neq 0 \qquad (2)$$

changes coordinate frames and coframes, and coordinate components of the geometrical objects in standard way.

In the introductory relativity textbooks [2] one usually says about coordinate transformations and about transformations of the coordinate components of the geometrical objects. In fact, it is sufficient. Also some conservative specialist on tensor analysis follow this way [3]. But one can use in **GR** (and in tensor calculus also) arbitrary frames, especially *non-holonomic* (or anholonomic) frames and coframes $(\{h_a{}^{\mu}(x)\}, \{h^b{}_{\alpha}(x)\})$: $h_a{}^{\mu}(x)h^b{}_{\mu}(x) = \delta_a^b$, $(a, b, c, d, ..., = 0, 1, 2, 3)$. Latin indices (= anholonomic indices) numerate vectors and covectors.

The anholonomic frames and coframes *are not connected with local coordinates*, e.g., they are neutral under coordinate transformations. Instead of we have

$$\partial_{\alpha} = h^b{}_{\alpha}(x)\partial_b, \quad dx^{\alpha} = h_a{}^{\alpha}(x)dx^a, \qquad (3)$$

or, equivalently,

$$\vec{e}_a := \partial_a = h_a{}^{\beta}(x)\partial_{\beta}, \quad \vartheta^b := dx^b = h^b{}_{\mu}(x)dx^{\mu}. \qquad (4)$$

Here $(x) := \{x^{\alpha}\}$ are *spacetime coordinates*, and $\{x^a\}$ mean *tangent space coordinates*. [3]

For coordinate frames and coframes one has

$$\vec{e}_a = \delta_a^{\beta}\partial_{\beta}, \quad \vartheta^b = \delta_{\mu}^b dx^{\mu}. \qquad (5)$$

Some remarks are in order:

1. $\{\vec{e}_a(x)\} \equiv \{\partial_a(x)\}$ is a coordinate frame in tangent space $T_x(M_4, g_L)$, and $\{\vartheta^b\} \equiv \{dx^b\}$ is a coordinate coframe in the dual space space $T_x^*(M_4, g_L)$.

   Differential forms $\vartheta^b = dx^b = h^b{}_{\mu}(x)dx^{\mu}$ *are not integrable* for anholonomic frames $\{h^b{}_{\mu}(x)\}$: $d\vartheta^b \neq 0$.

---

[3]In **GR** every tangent space is endowed with Minkowski structure.

3

2. Henceforth we will consequently use an old tensorial terminology of J.A. Schouten, and S. Gołąb, i.e., we will call $\{h_a{}^\beta(x)\}$ "frame" instead of $\{\vec{e}_a(x)\}$, and $\{h^b{}_\mu(x)\}$ "coframe" instead of $\{\vartheta^b\}$. It will useful in passing to teleparallel gravity because majority of the authors working in this field uses this terminology.

3. We permanently use standard Einstein summation convention.

As we see, anholonomic frames and coframes in our terminology *connect* the partial derivatives ∂_α and ∂_b, and differentials dx^α with dx^a. They also connect anholonomic components of the geometrical objects (denoted by Latin indices) with their coordinate components (denoted by Greek indices). Namely, one has (coordinates $\{x^\mu\}$ are fixed) for a tensor field of the type (r,s)

$$T^{a_1...a_r}{}_{b_1...b_s}(x) = h^{a_1}{}_{\mu_1}(x)...h^{a_r}{}_{\mu_r}(x)h_{b_1}{}^{\nu_1}(x)...h_{b_s}{}^{\nu_s}(x)T^{\mu_1...\mu_r}{}_{\nu_1...\nu_s}(x), \qquad (6)$$

and, conversely

$$T^{\mu_1...\mu_r}{}_{\nu_1...\nu_s}(x) = h_{a_1}{}^{\mu_1}(x)...h_{a_r}{}^{\mu_r}(x)h^{b_1}{}_{\nu_1}(x)...h^{b_s}{}_{\nu_s}(x)T^{a_1...a_r}{}_{b_1...b_s}(x). \qquad (7)$$

For a linear and metric connection ω one obtains[4]

$$\omega^a{}_{bc}(x) = h_c{}^\nu(x)\omega^a{}_{b\nu}(x), \qquad (8)$$

where

$$\omega^a{}_{b\nu}(x) = h^a{}_\lambda(x)\Gamma^\lambda{}_{\mu\nu}(x)h_b{}^\mu(x) + h^a{}_\rho(x)\partial_\nu h_b{}^\rho(x) \qquad (9)$$

is so-called *spin connection*. Conversely, we have

$$\Gamma^\rho{}_{\mu\nu}(x) = h_a{}^\rho(x)h^b{}_\mu(x)\omega^a{}_{b\nu}(x) + h_a{}^\rho(x)\partial_\nu h^a{}_\mu(x). \qquad (10)$$

In **GR** one usually uses the anholonomic frames $\{h_a{}^\mu(x)\}$ and dual coframes $\{h^b{}_\mu(x)\}$ which form the so-called *orthonormal tetrad and cotetrad* fields. These fields are defined as follows

$$h^a{}_\mu(x)h^b{}_\nu(x)\eta_{ab} = g_{\mu\nu}(x), \qquad (11)$$

or, equivalently

$$h_a{}^\mu(x)h_b{}^\nu(x)g_{\mu\nu}(x) = \eta_{ab}. \qquad (12)$$

---

[4]From here we confine to anholonomic tetrads and cotetrads (See below).

Here $\eta_{ab} = diag(1, -1, -1, -1)$ is the Minkowski metric of the tangent spaces $T_x(M_4, g_L)$ and $g_{\mu\nu}(x)$ means the spacetime metric g_L.

The transformations of the spacetimes coordinates act only on spacetime indices (Greek indices) in standard way, whereas on the tangent space indices (Latin indices) act only *local or global Lorentz transformations*, e.g.,

$$h'^a{}_\mu = \Lambda^a{}_b(x) h^b{}_\mu(x), \tag{13}$$

where

$$\Lambda^a{}_b(x) \eta_{ac} \Lambda^c{}_d(x) = \eta_{bd}. \tag{14}$$

For a global Lorentz transformation one has $\Lambda^a{}_b = const.$

Tetrads are not uniquely determined by the given spacetime metric $g_{\mu\nu}(x)$ but only up to local Lorentz transformations, i.e., up to six arbitrary functions. It is because a metric has only ten independent components and a tetrad field has sixteen independent components. So, for a given metric $g_{\mu\nu}(x)$ there exists ∞^6 different classes of tetrad fields $\{h_a{}^\mu(x)\}$ which satisfy (11)-(12)[5].

Contrary, given tetrad field $\{h_a{}^\mu(x)\}$ *determines unique metric*

$$g_{\mu\nu}(x) = h^a{}_\mu(x) h^b{}_\nu(x) \eta_{ab}, \tag{15}$$

where

$$h^a{}_\mu(x) h_b{}^\mu(x) = \delta^a_b. \tag{16}$$

In **GR** fundamental role plays the spacetime metric $g_{\mu\nu}(x)$ (it is an observable), whereas the orthonormal tetrads (they are not observables) play only an auxiliary role: they simplify calculations and they enable us to introduce spinors into spacetime structure.

The physical foundations and standard formulation of the **GR** [6] have very good observational evidence. Observational consequences of the Einstein equations were confirmed up to $0,003\%$ in Solar System (weak gravitational field), and up to $0,05\%$ in binary pulsars (strong gravitational field). Universality of the free falls was confirmed up to 10^{-14} and some other consequences of the **EEP** were confirmed up to 10^{-23} (See, e.g., [1].).

So, up to now, we *needn't modify or generalize* **GR**. (Ockham razor).

---

[5] One class of the tetrad $[\{h_a{}^\mu(x)\}]$ means these tetrads which are connected by a global Lorentz transformation.

[6] We mean here **EEP**, Einstein equations and mathematical model (M_4, g_L) of the physical spacetime.

We would like to emphasize that *we have no free parameter* in **GR**.

Fascinating is that despite this the theory has passed all the stringent tests with favour. [7]

2 Teleparallel gravity

This is a gravity with *an absolute parallelism*, i.e., with *curve independent parallelism* of distant vectors and tensors.

In this old approach (since 1928; renewed recently) the mathematical model of the physical spacetime is based on *Weitzenböck geometry* (= teleparallel geometry or geometry with absolute parallelism).

The geometry of such a kind *is uniquely determined* by the given tetrad field $\{h_a{}^\mu\}(x)$. Namely, one has (Coordinates $\{x^\alpha\}$ are fixed):

1. Metric $g_{\mu\nu}(x) := h^a{}_\mu(x)h^b{}_\nu(x)\eta_{ab}$.

2. Teleparellel connection (Weitzenböck's connection) $\Gamma^\rho{}_{\mu\nu} := h_a{}^\rho(x)\partial_\nu h^a{}_\mu(x)$.

Here $h_a{}^\mu(x)h^b{}_\mu(x) = \delta^b_a$.

The teleparallel Weitzenböck connection [8] has non-vanishing torsion $T^\rho{}_{\mu\nu} := \Gamma^\rho{}_{\nu\mu} - \Gamma^\rho{}_{\mu\nu}$ iff the tetrads $\{h_a{}^\mu(x)\}$ are anholonomic, and it has identically vanishing curvature $R^\rho{}_{\theta\mu\nu}(\Gamma)$, where

$$R^\rho{}_{\theta\mu\nu}(\Gamma) := \partial_\mu\Gamma^\rho{}_{\theta\nu} - \partial_\nu\Gamma^\rho{}_{\theta\mu} + \Gamma^\rho{}_{\sigma\mu}\Gamma^\sigma{}_{\theta\nu} - \Gamma^\rho{}_{\sigma\nu}\Gamma^\sigma{}_{\theta\mu}. \tag{17}$$

Important remarks are in order:

1. Weitzenböck connection *is metric*, i.e.,

$$\nabla_\rho g_{\mu\nu} := \partial_\rho g_{\mu\nu} - \Gamma^\alpha{}_{\mu\rho}g_{\alpha\nu} - \Gamma^\alpha{}_{\nu\rho}g_{\mu\alpha} \equiv 0.^9 \tag{19}$$

---

[7]In the proposed *generalized gravity theories* one has many free parameters, e.g., one has 28 free parameters in metric-affine gravity. These parameters can be adjusted in order to have agreement with experience.

[8]In the following we will call it "Weitzenböck connection".

[9]But the other possible covariant derivative

$$\tilde\nabla g_{\mu\nu}(x) := \partial_\rho g_{\mu\nu} - \Gamma^\alpha{}_{\rho\mu}g_{\alpha\nu} - \Gamma^\alpha{}_{\rho\nu}g_{\mu\alpha}, \tag{19}$$

is different from zero because Weitzenböck connection is not symmetric.

2. Torsion of the Weitzenböck connection *is entirely determined* by the Schouten-Van Danzig *anholonomy object* $\Omega^a{}_{bc}(x)$, where

$$\Omega^a{}_{bc}(x) := h_b{}^\beta(x)h_c{}^\gamma(x)[\partial_\gamma h^a{}_\beta(x) - \partial_\beta h^a{}_\gamma(x)].^{10} \tag{20}$$

Namely, we have

$$T^\rho{}_{\mu\nu}(x) = h_a{}^\rho(x)h^b{}_\mu(x)h^c{}_\nu(x)\Omega^a{}_{bc}(x). \tag{21}$$

3. One has the following relation between the components of the Weitzenböck connection $\Gamma^\rho{}_{\mu\nu}(x)$ and between the components $\{^\rho{}_{\mu\nu}\}(x)$ of the Levi-Civita connection for the metric $g_{\mu\nu}(x)$

$$\Gamma^\rho{}_{\mu\nu}(x) = \{^\rho{}_{\mu\nu}\}(x) + K^\rho{}_{\mu\nu}(x), \tag{22}$$

where

$$K^\rho{}_{\mu\nu}(x) := \frac{1}{2}(T_\mu{}^\rho{}_\nu + T_\nu{}^\rho{}_\mu - T^\rho{}_{\mu\nu}) \tag{23}$$

is the *contortion tensor*.

4. For Weitzenböck connection $\Gamma^\rho{}_{\mu\nu}(x)$

$$\omega^a{}_{b\nu}(x) \equiv 0 \Rightarrow \omega^a{}_{bc} \equiv 0, \tag{24}$$

i.e., this *connection identically vanishes* in the tetrads $\{h_a{}^\mu(x)\}$ which have determined it.

Greek, i.e., holonomic indices are raised and lowered with the spacetime metric $g_{\mu\nu}$ and the Latin, i.e., anholonomic indices, are raised and lowered with the Minkowski metric η_{ab}.

The class of the tetrads $[\{h_a{}^\mu(x)\}]$ connected by global Lorentz transformations with $\Lambda^a{}_b = const$ determines the same Weitzenböck connection and geometry. On the other hand, the any two tetrad fields $\{h'^a{}_\mu(x)\}$, $\{h^a{}_\mu(x)\}$ which are connected by a local Lorentz transformation

$$h'^a{}_\mu(x) = \Lambda^a{}_b(x)h^b{}_\mu(x) \tag{25}$$

determine two different Weitzenböck connections, $\bar{\Gamma}^\rho{}_{\mu\nu}(x)$ *and* $\Gamma^\rho{}_{\mu\nu}(x)$ *and two different Weitzenböck geometries.*

---

[10]The anholonomity object measures anholonomy of the used tetrad field: for a holonomic tetrads $\{h_a{}^\mu(x)\}$ one has $\Omega^a{}_{bc}(x) \equiv 0$.

So, the set of the all tetrads ($\{h_a{}^\mu(x)\}$) splits onto *disjoint classes* (∞^6 classes[11]) which determine *different Weitzenböck connections and geometries.*

In consequence, the symmetry group of a teleparallel gravity consists of the group **DiffM₄** and the global Lorentz group.

In the following we will confine to the very special case of the teleparallel gravity, namely we will confine to the so-called *teleparallel equivalent of general relativity* (**TEGR**).

The **TEGR** is a recent approach to teleparallel gravity which is mainly developed by mathematicians and physicists from Brasil (See, e.g., [4].).

One can look on **TEGR** as a new trial to rescue torsion in theory of gravity because, up to now, *no experiment confirmed the Riemann-Cartan torsion*[12].

The details of the standard approach to **TEGR** read.

One starts with the given metric $g_{\mu\nu}(x)$. This metric determines (up to local Lorentz transformations) the anholonomic tetrad $\{h_a{}^\mu(x)\}$ and dual cotetrad $\{h^a{}_\mu(x)\}$ fields, which satisfy

$$h^a{}_\mu(x)h^b{}_\nu(x)\eta_{ab} = g_{\mu\nu}(x), \tag{26}$$

$$h^a{}_\mu(x)h_b{}^\mu(x) = \delta^a_b. \tag{27}$$

Then, these fields determine the Weitzenböck connection

$$\Gamma^\rho{}_{\mu\nu}(x) = h_a{}^\rho(x)\partial_\nu h^a{}_\mu(x), \tag{28}$$

which satisfies

$$\{^\rho{}_{\mu\nu}\}(x) = \Gamma^\rho{}_{\mu\nu}(x) - K^\rho{}_{\mu\nu}(x). \tag{29}$$

Here $\{^\rho{}_{\mu\nu}\}(x)$ is the Levi-Civita connection for the metric $g_{\mu\nu}(x)$.

For the Weitzenböck connection $\Gamma^\rho{}_{\mu\nu}(x)$ one has

$$R^\rho{}_{\theta\mu\nu}(\Gamma) \equiv R^\rho{}_{\theta\mu\nu}(\{\}) + Q^\rho{}_{\theta\mu\nu} \equiv 0. \tag{30}$$

Here

$$R^\rho{}_{\theta\mu\nu}(\Gamma) := \partial_\mu\Gamma^\rho{}_{\theta\nu} - \partial_\nu\Gamma^\rho{}_{\theta\mu} + \Gamma^\rho{}_{\sigma\mu}\Gamma^\sigma{}_{\theta\nu} - \Gamma^\rho{}_{\sigma\nu}\Gamma^\sigma{}_{\theta\mu}, \tag{31}$$

---

[11]∞^6 classes because the local Lorentz transformations depend on six arbitrary functions.

[12]The Riemann-Cartan torsion is the torsion in the Riemann-Cartan geometry. This generalized metric geometry endowed with curvature and torsion was proposed by many authors since 1970 [5] as a geometric model of the physical spacetime. In our opinion lack of experimental evidence, many ambiguities to whose torsion leads, topological triviality of torsion and *Ockham razor* rather disqualify this model [6].

$$R^{\rho}{}_{\theta\mu\nu}(\{\}) := \partial_\mu \{^{\rho}{}_{\theta\nu}\} - \partial_\nu \{^{\rho}{}_{\theta\mu}\} + \{^{\rho}{}_{\sigma\mu}\}\{^{\sigma}{}_{\theta\nu}\} - \{^{\rho}{}_{\sigma\nu}\}\{^{\sigma}{}_{\theta\mu}\}, \qquad (32)$$

and

$$Q^{\rho}{}_{\theta\mu\nu} := D_\mu K^{\rho}{}_{\theta\nu} - D_\nu K^{\rho}{}_{\theta mu} + K^{\rho}{}_{\sigma\mu} K^{\sigma}{}_{\theta\nu} - K^{\rho}{}_{\sigma\nu} K^{\sigma}{}_{\theta\mu}. \qquad (33)$$

D_μ is the Levi-Civita covariant derivative expressed in terms of the Weitzenböck connection, i.e.,

$$D_\rho v^\mu := \partial_\rho v^\mu + (\Gamma^{\mu}{}_{\lambda\rho} - K^{\mu}{}_{\lambda\rho}) v^\lambda. \qquad (34)$$

$R^{\rho}{}_{\theta\mu\nu}(\Gamma)$ is the $main^{13}$ curvature tensor of the Weitzenböck geometry.

The Authors which work on **TEGR**, by use the fundamental formulas (26),(29),(30) of the Weitzenböck geometry, *rephrase*, step by step, all the formalism of the purely metric **GR** in terms of the Weitzenböck connection $\Gamma^{\rho}{}_{\mu\nu}(x)$ and its torsion $T^{\rho}{}_{\mu\nu}(x)$ (Mainly in terms of torsion).

For example:

1. The Einstein Lagrangian for **GR**

$$L_E = (-)\alpha\sqrt{|g|}R(\{\}) + \partial_\mu w^\mu, \qquad (35)$$

where $g := det[g_{\mu\nu}]$, and

$$w^\mu := \alpha\sqrt{|g|}(g^{\alpha\beta}\{^{\mu}{}_{\alpha\beta}\} + g^{\alpha\mu}\{^{\gamma}{}_{\alpha\gamma}\}) \qquad (36)$$

is rephrased to the form

$$\alpha h S^{\rho\mu\nu} T_{\rho\mu\nu} =: L_{TEGR}, \qquad (37)$$

where $h = det[h^{a}{}_{\mu}] = \sqrt{|g|},^{14}$ and

$$S^{\rho\mu\nu} = (-)S^{\rho\nu\mu} := \frac{1}{2}[K^{\mu\nu\rho} - g^{\rho\nu}T^{\alpha\mu}{}_\alpha + g^{\rho\mu}T^{\alpha\nu}{}_\alpha]. \qquad (38)$$

2. The vacuum Einstein equations

$$[R_\lambda{}^\rho(\{\}) - \frac{1}{2}\delta_\lambda{}^\rho R(\{\})]\sqrt{|g|} = 0 \qquad (39)$$

---

[13] Main curvature tensor because one can consider other curvatures in Weitzenböck geometry, e.g., Riemannian curvature [7].

[14] One obtains in fact ∞^6 different L_{TEGR} because L_{TEGR}, like L_E is invariant only under global Lorentz group. Despite that the field equations (39)-(40) are locally Lorentz invariant. We could get localy Lorentz invariant L_{TEGR} if we rephrased $L = (-)\alpha\sqrt{|g|}R(\{\})$.

are rephrased to the form

$$\partial_\sigma(hS_\lambda{}^{\sigma\rho}) - 4\alpha^{(-1)}(ht_\lambda{}^\rho) = 0, \tag{40}$$

where

$$t_\lambda{}^\rho = h^a{}_\lambda J_a{}^\rho + 4\alpha\Gamma^\mu{}_{\lambda\nu}S_\mu{}^{\nu\rho}, \tag{41}$$

and

$$J_a{}^\rho = (-)4\alpha h_a{}^\lambda S_\mu{}^{\nu\rho}T^\mu{}_{\nu\lambda} + 4\alpha h_a{}^\rho S^{\alpha\beta\gamma}T_{\alpha\beta\gamma}, \tag{42}$$

and so on.

$\alpha := \frac{c^4}{16\pi G}$.

Then, these authors call the obtained formal reformulation of **GR** in terms of the Weitzenböck geometry *the teleparallel equivalent of the general relativity* (**TEGR**) and conclude: "Gravitational interaction can be described *alternatively* in terms of curvature, as it is usually done in **GR**, or in terms of torsion, in which case we have the so-called teleparallel gravity. *Whether gravitation requires a curved or torsional spacetime, therefore, turns out to be a matter of convention*". They also assert that **TEGR** "is better than the original **GR**" because, e.g., "in **TEGR** one can separate gravity from inertia (on the connection level) and this separation reads"

$$\{^\alpha{}_{\beta\gamma}\} = \Gamma^\alpha{}_{\beta\gamma} - K^\alpha{}_{\beta\gamma}. \tag{43}$$

Following the authors which work on **TEGR**, the left hand side term of the above "separation formula", ($\{^\alpha{}_{\beta\gamma}\}$), represents gravity and inertia and the right hand side terms describe inertia, ($\Gamma^\alpha{}_{\beta\gamma}$), and gravitation, ($K^\alpha{}_{\beta\gamma}$), respectively.

Of course, such separation *contradicts* **EEP** and *is impossible* in standard formulation of the **GR**.

We *cannot agree with such statements*. In our opinion, the "teleparallel equivalent of **GR**" (What kind of equivalence?) is only *formal and geometrically trivial*, non-unique (See below) rephrase of **GR** in terms of the Weitzenböck geometry. Such rephrase is, of course, *always possible* not only with **GR** but also with any other purely metric theory of gravity.

In our opinion, *we have no profound physical motivation* for expression of the gravitational interaction in terms of the teleparallel torsion because the Weitzenböck torsion *is entirely expressed in terms of the Van Danzig*

10

and Schouten anholonomity object $\Omega^a{}_{bc}(x)$. So, the torsion of the teleparallel Weitzenböck connection *describes only anholonomy* of the used tetrad field and, therefore, *it is not connected neither with the real geometry of the physical spacetime nor with real gravity*, e.g., one can introduce Weitzenböck torsion already in flat Minkowski spacetime.

Weitzenböck torsion could only describe the inertial forces in the framework of the special relativity [15].

Contrary, the Levi-Civita part of the Weitzenböck connection, *as independent of tetrads*[16] , can have *and surely has* the physical and geometrical meaning.

Further ctitical remarks on **TEGR**.

1. **TEGR** is nothing new. In fact, it is exactly the old tetrad formulation of **GR** given in the very distant past by C. Møller [8] but expressed in terms of anholonomy of the tetrads instead of in terms of tetrads exclusively (As it was in Møller papers). For example, despite that the **TEGR** field equations are expressed in terms of torsion of the Weitzenböck geometry, they form the system of the 10 partial differential equations of the 2nd order on 16 tetrads components, like the 10 field equations of the Møller's tetrad formulation of **GR**. Solving the **TEGR** equations in vacuum (or in matter) we are looking for the tetrad components $\{h_a{}^\mu(x)\}$ for *apriori given general form of the metric* $g_{\mu\nu}(x)$; not for the components of torsion. Weitzenböck connection and its torsion are calculated later [9].

   Therefore, the notation of the Lagrangian and the field equations of **TEGR** in terms of Weitzenböck torsion *is only a camouflage*: **TEGR** is simply the Møller's tetrad formulation of **GR**, and, like Møller's formulation of **GR**, determines uniquely the metric only.

   We would like to emphasize that one can find all the results of the **TEGR** including the **TEGR** energy-momentum tensor for pure gravity in the old Møller's papers.[17]

2. **TEGR** *is not unique.* This follows from the fact: given metric, $g_{\mu\nu}(x)$

---

[15]In special relativity anholonomic tetrads really represent non-inertial frames.

[16]The Levi Civita connection depends only on metric. It is independent of the tetrads which determine the same spacetime metric.

[17]This 'tensor" is one of the most important results obtained in the framework of **TEGR**.

has 10 intrinsic components and determines only 10 components of the tetrads field $\{h_a{}^\mu(x)\}$ which has 16 intrinsic components. It is a consequence of the known fact that a given metric *determines tetrad field up to local Lorentz transformations*, which form the local, six-parameters, ortochronous Lorentz group $L_+^\uparrow$ defined as follows

$$
\begin{aligned}
L_+^\uparrow &= \{\Lambda^a{}_b(x) : \Lambda^a{}_b(x)\eta_{ac}\Lambda^c{}_d(x) = \eta_{bd}, \\
det[\Lambda^a{}_b(x)] &= 1, \quad \Lambda^0{}_0 \geq 1\}.
\end{aligned}
\tag{44}
$$

The ten field equations of **GR** (or **TEGR**) determine the metric and also determine only ten components of the tetrad field. The remaining six components are lefting arbitrary functions of the spacetime coordinates $\{x^\alpha\}$ and can be arbitrarily established. It is a consequence of the local Lorentz invariance of the **TEGR** and **GR** field equations.

So, for the given metric, $g_{\mu\nu}(x)$, (**GR**) there exist ∞^6 different classes of tetrad fields (**TEGR**) and, in consequence, ∞^6, different Weitzenböck connections $\Gamma^\rho{}_{\mu\nu}(x)$ (and geometries). Each of these connections satisfies the equations

$$
\{^\rho{}_{\mu\nu}\}(x) = \Gamma^\rho{}_{\mu\nu}(x) - K^\rho{}_{\mu\nu}(x).
\tag{45}
$$

In the above equations the left hand side *is independent of tetrads*; it depends only on metric $g_{\mu\nu}(x)$, whereas the both terms on the right hand side *depend on the class of the tetrads*[18].

As a result we obtain ∞^6 different Lagrangians (37) for **TEGR** and ∞^6 different **TEGR**. This fact was already known C. Møller in context of his tetrad formulation of **GR**. Namely, Møller, in fact, also has obtained ∞^6 different tetrad formulations of **GR** because, the 10 field equations of his tetrad formulation of **GR**, identical with Einstein equations (1), determine the tetrad field up to local Lorentz transformations, i.e., up to six arbitrary functions. These field equations determine the metric only.[19]. In order to have field equations which would determine tetrad

---

[18] (One) class of tetrads := the set of tetrads $\left[\{h_a{}^\mu(x)\}\right]$ which are connected by global Lorentz transformations. Class of tetrads determines the same Weitzenböck connection and geometry. Different classes of tetrads are connected by local Lorentz transformations and determine different Weitzenböck connections and geometries.

[19] The same situation we have of course in the framework of the **TEGR** because the 10 field equations (40), like Møller's equations, are locally Lorentz invariant.

field completely (apart from constant Lorentz rotations) Møller has developed *tetrad theory of gravity* in which one has sixteen field equations onto sixteen tetrad components.

3. The authors which work on **TEGR** assert that the formula (43) (or (45)) gives *separation* of inertia ($\Gamma^\rho{}_{\mu\nu}(x)$) from gravity ($K^\rho{}_{\mu\nu}(x)$).

   Such speculative separation allows them, among other things, to introduce an energy-momentum tensor for gravity [20]. But this separation *is illusoric* because there exist ∞^6 different separations of the form (43) (or (45)) for given $\{^\alpha{}_{\beta\gamma}\}$, i.e., we *have no separation inertia from gravity* in **TEGR** (In agreement with **EEP**).

   In consequence, *we have no unique gravitational energy-momentum tensor* in **TEGR**.

4. The experts on **TEGR** transform trivially the geodesic equations of **GR**
$$\frac{d^2x^\alpha}{ds^2} + \{^\alpha{}_{\beta\gamma}\}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = 0 \tag{46}$$
   onto the *forces equations*
$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = K^\alpha{}_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} \tag{47}$$
   by putting in (46)
$$\{^\alpha{}_{\beta\gamma}\} = \Gamma^\alpha{}_{\beta\gamma} - K^\alpha{}_{\beta\gamma}. \tag{48}$$

   The *forces equations* (47) remind the **GR** equations of motion for a charged test particle when the both fields, electromagnetic and gravitational, simultaneously act on the particle
$$\frac{d^2x^\alpha}{ds^2} + \{^\alpha{}_{\beta\gamma}\}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = \frac{Q}{m}F^\alpha{}_\beta\frac{dx^\beta}{ds}. \tag{49}$$

   Here Q, m denote electric charge and mass of the particle respectively and $F^\alpha{}_\beta$ mean electromagnetic field acting on the particle. [21]

---

[20]It is in fact a family of ∞^6 different tensors the same as the family of the tensors which has been obtained many years ago by C. Møller *without any separation* in his tetrad formulation of **GR**.

[21]The right hand side of (49) is the electromagnetic force per unit mass which acts on the particle.

The specialists on **TEGR** try to attach some physical meaning to the *force equations* (47), namely following them, the right hand side of (47) describes *gravitational force* acting on the particle, whereas the term $\Gamma^{\alpha}{}_{\beta\gamma}\frac{dx^{\beta}}{ds}\frac{dx^{\gamma}}{ds}$ describes *inertial force.*

But there exist ∞^6 different reformulatios of the geodesic equations (46) to the form (47) with different $\Gamma^{\alpha}{}_{\beta\gamma}$ and $K^{\alpha}{}_{\beta\gamma}$. Which one of them is correct, i.e., *which one of them gives correct inertial force and correct gravitational force*?

Talking about *equivalence* of **TEGR** with **GR** *is misleading* because there exist ∞^6 different **TEGR** in consequence of the local Lorentz invariance of the field equations (40) [22].

Here we have the same kind of *"equivalence"* as the *"equivalence"* between a given metric $g_{\mu\nu}(x)$ (10 functions) and a tetrad field (16 functions), which satisfies $h^a{}_{\mu}(x)h^b{}_{\nu}(x)\eta_{ab} = g_{\mu\nu}(x)$ i.e., *we have no equivalence* [23].

Incorrect is also statement of the specialists on **TEGR** that Weitzenböck geometry is flat, like Minkowski geometry. In fact, e.g., *Riemannian curvature of such geometry is non-zero*. Also the curvature tensor $\tilde{R}^{\alpha}{}_{\beta\gamma\delta}(\Gamma)$ where

$$\tilde{R}^{\alpha}{}_{\beta\gamma\delta}(\Gamma) := \partial_{\gamma}\Gamma^{\alpha}{}_{\delta\beta} - \partial_{\delta}\Gamma^{\alpha}{}_{\gamma\beta} + \Gamma^{\alpha}{}_{\gamma\sigma}\Gamma^{\sigma}{}_{\delta\beta} - \Gamma^{\alpha}{}_{\delta\sigma}\Gamma^{\sigma}{}_{\gamma\beta} \tag{50}$$

is different from zero.

The tensor $\tilde{R}^{\alpha}{}_{\beta\gamma\delta}(\Gamma)$ differs from the former *main curvature tensor* $R^{\alpha}{}_{\beta\gamma\delta}(\Gamma)$ (See the formula (31)) by transposition lower indices in $\Gamma^{\alpha}{}_{\beta\gamma}(x)$.[24]

Resuming, in our opinion, **TEGR** is nothing new. It is camouflaged, the very old tetrad formulation of **GR** given by C. Møller, and it, *by no means is better* than standard **GR**. Contrary, standard **GR** *is surely better* than any **TEGR** because **GR** is invariand under any change of tetrads, whereas **TEGR** is not. **TEGR**, like any teleparallel gravity, is invariant only under global Lorentz rotations of tetrads.

We will finish with some general remarks about teleparallel gravity.

It should be emphasized that there exist many other approaches to teleparallel gravity, different from **TEGR**, and which generalize **GR**. At the

---

[22]But we must emphasize that every **TEGR** determines unique and the same metric structure of the spacetime as **GR** does. So, from the metric point of view, the different **TEGR** are equivalent.

[23]Remark also that metric and tetrads are different geometric objects.

[24]For Riemannian geometry, owing to symmetry of the Levi-Civita connection, these both tensors are identically equal.

first time such approach to gravity was considered already by A. Einstein ("Fernparallelismus" in 1928 [10]) and then by C. Møller (1978), Pellegrini and Plebanski [11], Hayashi and Shirafuji [12], and others. Recently the teleparallel approach to gravity is developed by F.B. Estabrook, Y. Itin, and L. Schücking [13].

In these other approaches to teleparallel gravity the gravitational Lagrangian is built from irreducible torsion componets or from tetrads immediately, and contains, in general, three free parameters to be determined by experiments. This Lagrangian is invariant under **DiffM$_4$** and has also global Lorentz symmetry.

The fundamental geometric object are tetrads which determine spacetime metric and Weitzenböck connection, and, therefore, all the local Weitzenböck geometry of the physical spacetime.

In vacuum, we have in these approaches sixteen 2nd order field equations on sixteen tetrad components. The field equations should determine the tetrads field $h_a{}^\mu(x)$ up to constant Lorentz rotations, i.e., up to global Lorentz group, and owing that, should determine a unique Weitzenböck geometry. But tetrads *are not observables*: they are very alike to the electromagnetic potentials. Moreover, there are problems with physical interpretation of the six additional tetrads components (10 components can describe gravitational field, but what about remaining 6 components?) and these theories suffer from badly posed Cauchy problem [14].

References

[1] C.M. Will, "Theory and Experiment in Gravitational Physics", Cambridge University Press, Cambridge 1993; "The Confrontation between General Relativity and Experiment", arXiv:gr-qc/0510072.

[2] B.F. Schutz, "A First Course in General Relativity", Cambridge University Press, Cambridge 1985 (Polish edition: B.F. Schutz, "Wstęp do Ogólnej Teorii Względności", Wydawnictwo Naukowe PWN, Warszawa 2002); L.D. Landau, E.M. Lifshitz, "The Classical Theory of Fields", 4th edition, Oxford 2002 (Polish edition: L.D. Landau, E.M. Lifszyc, "Teoria Pola", Wydawnictwo Naukowe PWN, Warszawa 2009); J. Foster, J.D. Nightingale,"A Short Course in General Relativity", Longman, London and New York 1979 (Polish edition: J. Foster, J.D. Nightingale, Ogólna Teoria Względności", PWN, Warszawa 1985); J. Plebański and A. Krasiński, "An Introduction to General Relativity and Cosmology", Cambridge University Press, Cambridge 2006; S. Carrol, "Spacetime and Geometry. An Introduction to General Relativity", Addison Wesley, 2004; W. Rindler, "Relativity, Special, General, and Cosmological", Oxford University Press, Oxford 2004; A.Trautman, W. Kopczyński, "Czasoprzestrzeń i Grawitacja", PWN, Warszawa 1984 (There exists English edition).

[3] J.A. Schouten, "Ricci–Calculus", Springer-Verlag, Berlin 1954; S. Gołąb, "Tensor Calculus", PWN, Warsaw 1966 (In Polish.There exists English edition.).

[4] V.C. de Andrade, L.C.T. Guillen and J.G. Pereira, *Int. J. Mod. Phys.*, **D13**(2004) 2193 (arXiv:gr-qc/0501017); R. Aldrovandi, T.G. Lucas and J.G. Pereira, "Does a tensorial energy-momentum density for gravitation exist?", arXiv:0812.0034 [gr-qc]; R. Aldrovandi, T.G. Lucas and J.G. Pereira, "Inertia and gravitation in teleparallel gravity", arXiv:0812.0034 v.2 [gr-qc]; V.C. de Andrade, H.I. Arcos and J.G. Perei ra, "Torsion as Alternative to Curvature in the Description of Gravitation", arXiv: gr-qc/0412034; H.I. Arcos, T.G. Lucas and J.G. Pereira, "A Consistent Gravitationally–Coupled Spin–2Field Theory", arXiv:1001.3407 [gr-qc]; R.Aldrovandi, J.G. Pereira and K.H. Vu, "Doing without the Equivalence principle", arXiv:gr-qc/0410042; R.A. Mosua and J.G. Pereira, *Gen. Rel. Gravit.*, **36** (2004) 2525 (arXiv:gr-qc/0312093); R. Aldrovandi and J.G. Pereira, "Gravitation: in search of the missing torsion", arXiv:0801.4148 [gr-qc]; H.I. Arcos and J.G. Pereira, "Torsion and the gravitational interaction", arXiv:gr-qc/0408096; J.G. Pereira, T. Vargas and C.M. Zhang, " Axial–Vector Torsion and the Teleparallel Kerr Spacetime",

arXiv:gr-qc/0102070; V.C. de Andrade, L.C.T. Guillen and J.G. Pereira, "Teleparallel gravity: an overview", arXiv:gr-qc/0011087; T.G. Lucas, Y.N. Obukhov and J.G. Pereira, " Regularizing role of teleparallelism", arXiv:0909:2418 [gr-qc]; H.I. Arcos, V.C. de Andrade and J.G. Pereira, " Torsion and Gravitation: a New View", arXiv:gr-qc/0403074; J.W. Maluf, F.F. Faria and K.H. Castello-Branco, *Class. Quantum Grav.*, **20** (2003) 4683; V. C. de Andrade, L.C.T. Guillen and J.G. Pereira, "Teleparellel Spin Connection", arXiv:gr-qc/0104103; Y.N. Obukhov and J.G. Pereira, *Phys. Rev.*, **D67** (2003) 044008; A.A. Sousa, R.B. Pereira and A.C. Silva, "Energy and angular momentum densities in a Gödel–type universe in the teleparallel geometry", arXiv:0803.1481 [gr-qc]; J.F. da Rocha-Neto and K.H. Castello-Branco, "Gravitational Energy of Kerr and Kerr Anti-de Sitter Space-Times in the Teleparallel Geometry", arXiv:gr-qc/0205028; J.W. Maluf, J.F. da Rocha-Neto, T.M.L. Toribio and K.H. Castello-Branco, "Energy and angular momentum of the gravitational field in the teleparallel geometry", arXiv:gr-qc/0204035; M. Sharif and A.Jawal, "Energy Content of Some Well-Known Solutions in Teleparallel Gravity", arXiv:1005.5203 [gr-qc]; R. Aldrovandi, J.G. Pereira and K.H. Vu, "Selected Topics in Teleparallel Gravity", arXiv:gr-qc/0312008; R. Aldrovandi, P.B. Barros and J.G. Pereira, *Gen. Rel. Gravit.*, **35** (2003) 991; V.C. de Andrade, L.C. T. Guillen and J.G. Pereira, "Teleparallel Gravity and the Gravitational Energy-Momentum Density", arXiv:gr-qc/0011079; R. Aldrovandi and J.G Pereira, "An introduction to teleparallel gravity", Instituto de Fisica Teorica, UNESP, Sauo Paulo, Brasil 2007.

[5] A. Trautman, "On the Structure of the Einstein-Cartan Equations", Istituto Nazionale di Alta Matematica, *Symposia Mathematica*, **12** (1973) 139; F.W. Hehl, *Gen. Rel. Gravit.*, **4** (1973) 333; F.W. Hehl, –ibidem **5** (1974) 491; F.W. Hehl et al., *Rev. Mod. Phys.*, **48** (1976) 393; F.W. Hehl et al., "Gravitation and the Poincaré Gauge Field Theory with Quadratic Lagrangian", in *General Relativity and Gravitation*, Vol.I., Ed.A. Held, Plenum Publishing Corporation 1980; A. Trautman, "Fiber Bundles, Gauge Fields and Gravitation", in *General Relativity and Gravitation*, Vol.I. Ed. A. Held, Plenum Publishing Corporation 1980; A. Trautman, "Einstein-Cartan Theory", arXiv:gr-qc/0606062.

[6] J. Garecki, "On Torsion in a Theory of Gravity", in *Relativity, Gravi-*

17

tation, Cosmology, Eds. V. Dvoeglazov and A. Espinoza Garrido, 2004 Nova Science Publishers, Inc.

[7] M.I. Wanas, "Absolute Parallelism Geometry: Developments, Applications and Problems", arXiv:gr-qc/0209050.

[8] C. Møller, "Conservation laws in the tetrad theory of gravitation", in *Relativistic Theories of Gravitation*, Ed. L. Infeld, Pergamon Press, Oxford● London● Edinburgh● New York● Paris● Frankfurt. Copyright 1964 by PWN, Warszawa 1964; C. Møller, *Mat. Fys. Medd. Dan. Vid. Selsk.*, **35** (1966) 14pp (NORDITA publications No.190); C. Møller, "On the Crisis in the Theory of Gravitation and a Possible Solution", *Mat. Fys. Medd. Dan. Vid. Selsk.*, **39** (1978) 31pp, København 1978.

[9] G. Zet, " Schwarzschild Solution on a Space-Time with Torsion", arXiv:gr-qc/0308078.

[10] T. Sauer, 'Field equations in teleparallel spacetime:Einstein's *Fernparallelismus* approach towards unified field theory", arXiv:physics/0405142.

[11] J. Plebański,"Tetrads and Conservation Laws", in *Relativistic Theories of Gravitation*, Ed.L. Infeld, Pergamon Press, Oxford●London●Edinburgh●New York●Paris●Frankfurt. Copyright 1964 by PWN, Warszawa 1964.

[12] K. Hayashi and T. Shirafuji, *Phys. Rev.*, **D 19** (1979) 3524; F. Müller-Hoissen, "On the tetrad theory of gravity", MPI–PAE/Pth61/84, 17pp.

[13] Y. Itin, "Coframe geometry and gravity", arXiv:0711.4209 [gr-qc]; Y. Itin, "Does the coframe geometry can serve as a unification scheme?", arXiv:gr-qc/0409071; E.L. Schücking, "Einstein's Apple and Relativity's Gravitational Field", arXiv:0903.3768 [physics.hist-ph]; F.B. Estabrook, "Conservation Laws for Vacuum Tetrad Gravity", arXiv:gr-qc/0508081.

[14] W. Kopczyński, *J.Phys.*,**A 15** (1982) 493.

Teleparalelny ekwiwalent ogólnej teorii względności: uwagi krytyczne

Janusz Garecki

Instytut Matematyki Uniwersytetu Szczecińskiego

Streszczenie

Po przedstawieniu podstawowych faktów z ogólnej teorii względności oraz z teleparalelnej grawitacji, ograniczam się do analizy specjalnego modelu teleparalelnej grawitacji nazwanego przez jego twórców *teleparalelnym ekwiwalentem ogólnej teorii względności* (w skrócie **TEGR**). Model ten był(i jest) ostatnio intensywnie badany głównie przez matematyków i fizyków z Brazylii.

W pracy pokazuję, że **TEGR** jest zakamuflowanym, starym, tetradowym sformułowaniem ogólnej teorii względności, dokonanym w latach 60-tych i 70-tych XX-go wieku przez C. Møllera i podkreślam, że **TEGR** jest niejednoznacznym i trywialnym przeformułowanie ogólnej teorii względności, które nie może dać nic lepszego od standardowe sformułowanie tej teorii (Moim zdaniem, przeformułowanie to jest gorsze).